Rotating Electric Classical Solutions of $2+1$ D $U(1)$ Einstein Maxwell Chern-Simons

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Abstract. We study electric stationary radial symmetric classical solutions of the $U(1)$ Einstein Maxwell Chern-Simons theory coupled to a gravitational massless scalar field with a cosmological constant in $2+1$ dimensions. Generic aspects of the theory are discussed at an introductory level. We study solutions for both negative sign (standard) and positive sign (ghost) of the gauge sector concluding that although the expressions for the solutions are the same, the constants as well as the physics change significantly. A rotating electric point particle is found. For the standard sign and specific values of the parameters corresponding to solutions with positive mass the singularity is dressed (in the sense that itself constitutes an horizon). The space-time curvatures can be both positive or negative depending on the dominance of the scalar or topologically massive matter. The Chern-Simons term is responsible for interesting features, besides only allowing for rotating solutions, it imposes restrictive bounds on the cosmological constant Λ such that it belongs to a positive interval and is switch on and off by the topological mass m^2. Furthermore the charge, angular momentum and mass of the particle solution are expressed uniquely as functions of the ratio between the cosmological constant and the topological mass squared $x = \Lambda/m^2$. The main drawback of our particle solution is that the mass is divergent. Our background is a rotating flat space without angular deficit. We briefly discuss parity and time-inversion violation by the Chern-Simons term which is explicit in the solutions obtained, their angular momentum only depends on the relative sign between the Chern-Simons term and the Maxwell term. Trivial solutions are briefly studied holding non-singular extended configurations.

PACS numbers: 02.40.-k, 03.50.-z, 04.20Jb

1. Introduction

Several works have studied three dimensional classical gravitational configurations on topological and non topological field theories. The first works addressed Einstein theories, the well known AdS BTZ black hole [1], Einstein Maxwell Chern-Simons theory [2, 3] and rotating BTZ [4] (see also [5]).

This work studies the classical solutions for a $2+1$D Einstein Maxwell Chern-Simons theory coupled to a gravitational massless scalar field (that is often interpreted as a dilaton field in string-frame). It therefore extends the work already done, both in Einstein Maxwell Chern-Simons theories [6, 7, 8, 9, 10, 11], Einstein Maxwell theories [12, 13], Einstein Maxwell theories with dilatonic potentials [14, 15] and the more recent Dilaton Einstein Maxwell theories [16, 17, 18, 19, 20, 21].

Here we exclusively address pure electric solutions of $3D$ Einstein Maxwell Chern-Simons coupled to a massless scalar field. We try to present in a pedagogical way both general results and details of calculations. In particular our action resembles closely the one of the works [14, 15] together with a Chern-Simons term. However we start from a more generic action only particularizing the action due to the inexistence of other possible solutions.

The motivation to study our enlarged theory is two folded: the quantum consistence of the theory, and the embedding of a $3D$ system in a $4D$ world. First demanding quantum consistence of the theory we have to consider the Maxwell-Chern-Simons theory. Neither the pure Maxwell theory, neither the Chern-Simons theory are consistent at quantum level. If we start just with a Maxwell action, radiative (quantum) corrections will induce the Chern-Simons term and if we start with just a Chern-Simons action, quantum corrections will induce a Maxwell term, this correction is exact to all orders [22, 23] (see also [24] for a review). Secondly our world is $4D$, therefore by counting degrees of freedom we need a gravitational scalar field in a $3D$ physical systems. Although several ways to embed $2+1$ dimensional systems in $3+1$ dimensions, the existence of a gravitational massless scalar field is rather well established. Considering a dimensional reduction scheme we obtain what is called Dilaton [25, 26]. Alternatively one can consider the gauging under some symmetry that effectively reduces the dimensionality of the problem, this is the example of the massless scalar field of the works on polarized cylindrical gravitational waves in $3+1$ gravity [27, 28, 29, 30, 31]. It is not clear that this scalar field can always be interpreted as a dilaton field although for some particular actions it can be proved that it correspond to dilaton in string frame [14, 15].

We also note that most of the literature in Abelian gauge Chern-Simons address (anti-)self-dual solutions. Here we address pure electric solutions.

The article is organized in the following way. In section 2 we present and discuss generic results of the Einstein Maxwell Chern-Simons theory coupled to a scalar field. First we introduce and justify the Action. From it we derive the equations of motion and choose a suitable metric parameterization. Also we derive the charge, angular

momentum and the mass in the ADM formalism. In section 3 we solve the equations of motion in the Cartan-frame. In section 4 we compute the curvature, investigate the existence of singularities and horizons. Then in section 5 we compute the charge, angular momentum and mass for the configurations obtained. Finally in section 6 we summarize the solutions obtained and discuss them. In appendix Appendix A we introduce the Cartan Frame formalism (also known as non-coordinate frame) and derive the equations of motion and other useful formulae.

2. General Results

2.1. Action and EOM

We take a generic $2+1$D Einstein Gravity coupled to a massless scalar field with a Gauge Sector described by $U(1)$ Maxwell-Chern-Simons

$$\begin{aligned} S = & \frac{1}{2\pi}\int_M d^3x \left\{\sqrt{-g}\left[e^{a\phi}\left(R + 2\lambda(\partial\phi)^2\right) - e^{b\phi}\Lambda \right.\right. \\ & \left.\left. +\hat{\epsilon}\frac{e^{c\phi}}{2}\left(F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right)\right] - \hat{\epsilon}\frac{m}{2}\,\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}\right\} \end{aligned} \tag{2.1}$$

where a, b, c, λ and the cosmological constant Λ are numerical parameters of the theory. $\hat{\epsilon} = \pm 1$ simply sets the relative sign between the gauge sector and the gravitational sector.

Varying this action in relation to the fields A_μ, $g^{\mu\nu}$ and ϕ we obtain the equations of motion, i.e. the Maxwell, Einstein and scalar field equations

$$\begin{aligned} \partial_\alpha(\sqrt{-g}e^{c\phi}F^{\alpha\mu}) + \frac{m}{2}\,\epsilon^{\mu\alpha\beta}F_{\alpha\beta} &= \sqrt{-g}e^{c\phi}J^\mu \\ G_{\mu\nu} - a\nabla_\mu\partial_\nu\phi + a\,g_{\mu\nu}\nabla^2\phi + (\lambda - a^2)\partial_\mu\phi\partial_\nu\phi & \\ -\left(\frac{\lambda}{2} - a^2\right)g_{\mu\nu}(\partial\phi)^2 + \frac{1}{2}e^{(b-a)\phi}g_{\mu\nu}\Lambda &= 2e^{(c-a)\phi}T_{\mu\nu} \\ e^{a\phi}\left[2(2a^2-\lambda)\nabla^2\phi + 2a(2a^2-\lambda)(\partial\phi)^2\right] + (3a-b)e^{b\phi}\Lambda &= \hat{\epsilon}(a+c)e^{c\phi}F^2 \end{aligned} \tag{2.2}$$

where the Einstein and Stress-Energy tensors are defined as

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ T_{\mu\nu} &= \hat{\epsilon}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right) \end{aligned} \tag{2.3}$$

and the covariant derivative and Laplacian are as usual

$$\begin{aligned} \nabla_\mu\partial_\nu\phi &= \partial_\mu\partial_\nu\phi - \Gamma^\alpha{}_{\mu\nu}\partial_\alpha\phi \\ \nabla^2\phi &= \partial_\alpha\partial^\alpha\phi + \Gamma^\alpha{}_{\alpha\beta}\partial^\beta\phi \end{aligned} \tag{2.4}$$

Note that the scalar field equations presented are obtained from the usual variation of the action with respect to ϕ

$$e^{a\phi}\left[a\,R - 2\lambda\nabla^2\phi - a\lambda(\partial\phi)^2\right] - be^{b\phi}\Lambda = \hat{\epsilon}\,c\,e^{c\phi}F^2 \tag{2.5}$$

summed with the contraction of the 3 Einstein equations with the metric times $2a$. In this way the gravitational curvature is not present in equation (2.2).

Our convention for the Ricci tensor is

$$R_{\mu\nu} = -\Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} \tag{2.6}$$

we note that when considering a cosmological constant Λ the symmetric definition of the Ricci tensor, maintaining the same metric signature, is not equivalent and will account for the opposite sign of Λ. In order to justify this choice, in the next subsection, we give the example of 3-dimensional deSitter space, a known and well studied example with $\Lambda > 0$.

2.2. Metric, Ricci Tensor and Maxwell Tensor

We take several parameterizations of a radial symmetric metric, in polar coordinates $x^0 = t$, $x^1 = r$ and $x^2 = \varphi$ of the form

$$ds^2 = g_{tt}dt^2 + dr^2 + g_{\varphi\varphi}d\varphi^2 + 2g_{t\varphi}dtd\varphi \tag{2.7}$$

with signature $(-,+,+)$.

The Antisymmetric tensor has only the non vanishing components

$$F_{tr} = E_* \qquad F_{r\varphi} = B_* \tag{2.8}$$

All the functions g_{tt}, $g_{\varphi\varphi}$, $g_{t\varphi}$, E_*, B_* and ϕ are radial symmetric, i.e. are r dependent only.

There is a couple of important well establish points to stress to fully justify this ansatz.

The motivation of introducing the $g_{t\varphi}$ component of the metric is due to the Maxwell equations, in the presence of the Chern-Simons term (without external currents), not allowing for solutions $B_* = 0$ or $E_* = 0$ when $g_{t\varphi} = 0$ [2] (both must be null or both must be present). So when there is a Chern-Simons term in the action and we are considering only Electric or only Magnetic fields, we must have $g_{t\varphi} \neq 0$, otherwise both fields are null. In physical terms means that the space-time is rotating, although it can still be stationary as long as $g_{t\varphi}$ does not depend on the time coordinate.

Also one may consider a non null $F_{t\varphi}$ but for the metric parameterizations considered here the Maxwell Equation in (2.2) for $\mu = 1$ imposes it to be null.

Finally it is important to stress that one can add a generic parameterization for $g_{rr} = 1/L^2$ by introducing a new radial coordinate ρ such that $d\rho/dr = L$. This accounts for a choice of coordinates and therefore does not change the physical results presented here.

Although in $4D$ space-time the choice of metric (most positive or most negative diagonal) is not relevant, in $3D$ space-time one needs extra care in the relative definitions

between the metric and remaining tensor fields. The reader may also note that depending on the choice of $3D$ Minkowski metric the determinant is positive (for most negative diagonal) or negative (for most positive diagonal). In (2.7) we choose the last case to maintain the determinant of the metric negative. To justify the choice of the Ricci tensor (2.6) and clear any confusions concerning its definition we present a simple pedagogical example of the well known dS geometry which has positive cosmological constant. We consider a cosmological Einstein action

$$S_E = \int d^3x\sqrt{-g}\,(R-2\Lambda) \tag{2.9}$$

and a dS metric for an observer at r=0 corresponding to a cosmological constant $\Lambda = +1$, of the form [33]

$$ds^2 = -(1-r^2)dt^2 + \frac{1}{(1-r^2)}dr^2 + r^2 d\varphi^2 \tag{2.10}$$

with signature $(-,+,+)$ near the origin (where the observer is) and determinant $|g| = -r^2$. Varying the action with respect to $g^{\mu\nu}$ we obtain the well know equations of motion

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0 \tag{2.11}$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the usual Einstein tensor. For the given metric, computing explicitly the einstein tensor, we obtain $G_{00} = 1-r^2$, $G_{11} = -1/(1-r^2)$ and $G_{22} = -r^2$. This reads

$$G_{\mu\nu} = -g_{\mu\nu} \tag{2.12}$$

Therefore the cosmological constant is uniquely define trough the equations of motion as $\Lambda = +1$. Maintaining the metric signature and the action and considering the symmetric definition of the Ricci tensor $\tilde{R}_{\mu\nu} = -R_{\mu\nu}$ we would obtain $\tilde{G}_{\mu\nu} = -G_{\mu\nu}$ and hence $\Lambda = -1$. Together with the definition $\tilde{R}$, if we swap the signature of the metric to $(+,-,-)$ maintaining the action or if we maintain the signature of the metric and change the action to $\tilde{S}_E = \int (R+2\Lambda)$ we would obtain $\Lambda = +1$. Also using the definition R, swapping the signature of the metric to $(+,-,-)$ and considering the action $\tilde{S}_E$ we would obtain $\Lambda = +1$.

So we conclude that the choices of the definition of the Ricci tensor, the metric signature and the relative sign of the cosmological constant and the gravitational curvature in the action are not all equivalent. Resuming, we choose the definition of the Ricci tensor given by (2.6), the metric signature $(-,+,+)$ and an action of the form (2.1).

Finally we briefly discuss the relative sign between the several terms in the action. First we note that we consider opposite signs between the Chern-Simons term the Maxwell term. This is to ensure that the photon mass is real $(\nabla^2 - m^2)F^* = 0$ [34, 35], if they have the same sign we would obtain imaginary (tachyonic) masses. In particular this choice sets the sign of the angular momentum J, as we will se our solutions have $J \sim m$ (or $\operatorname{sign}(m)$). This is an effect of parity violation and is expected because the

Chern-Simons term violates parity in the gauge sector. If we change the relative sign between the Maxwell term and the Chern-Simons term the only effect on the solutions is to change the sign of the angular momentum. However as we explained this accounts for the photon to become a tachyon, for this reason we fixed this choice.

$\hat{\epsilon} = \pm 1$ sets the relative sign between the gauge sector (Maxwell term F^2) and the Einstein term (R). Choosing $\hat{\epsilon} = +1$ or $\hat{\epsilon} = -1$ does not change the expressions for the solutions, nevertheless the validity range for the parameters will change significantly, therefore the physical interpretation of the results as well. Also it is interesting to note that upon quantization the sign of the Maxwell is relevant. If we have $\hat{\epsilon} = -1$ we obtain the standard Hamiltonian and excited states of the gauge fields will have positive energy for Bose-Einstein spin-statistics, while for $\hat{\epsilon} = +1$ the excited states for the gauge fields will only hold positive energy for Fermi-Dirac spin-statistics. In this case the gauge fields have the *wrong* spin-statistics and for that reason are commonly called ghosts. It is quite interesting that different choice of signs will also at classical level hold significant differences as we will see in detail.

2.3. Mass, Charge and Angular Momentum

We are going to use the ADM formalism [36] (see [37]), so we rewrite the line element using a generic parameterization‡

$$ds^2 = -f^2 dt^2 + dr^2 + h^2(d\varphi + A dt)^2 \tag{2.13}$$

and considering the Hamiltonian form of the action

$$S = -2\pi\Delta t \int dr\,[-f\mathcal{H} + A\mathcal{H}^{\varphi} + A_0\mathcal{G}] + S_{\mathcal{B}} \tag{2.14}$$

where $S_{\mathcal{B}}$ stands for boundary terms due to the integration by parts of the terms containing f', f'', A' and A_0'

$$S_{\mathcal{B}} = \frac{1}{2\pi}\int_{\partial M} d^2x \left[f\,2e^{a\phi}(2a\,h\,\phi' + h') + A\,e^{a\phi}\Pi_G + \hat{\epsilon}A_0\left(\Pi_{EM} - \frac{m}{2}A_{\varphi}\right)\right] \tag{2.15}$$

and the Hamiltonian, Momentum and Gauss constraints are respectively

$$\begin{aligned}
\mathcal{H} &= -\frac{2\Pi_G^2}{h^3}e^{a\phi} - 2a\left(h\phi' e^{a\phi}\right)' - 2h'' e^{a\phi} + 2\lambda h(\phi')^2 e^{a\phi} + \Lambda\,h\,e^{b\phi} \\
&+ \hat{\epsilon}\left(\frac{e^{-c\phi}}{h}\left(\Pi_{EM} + \frac{m}{2}A_{\varphi}\right)^2 + h\,e^{c\phi}\,(A_{\varphi}')^2\right) \\
\mathcal{H}^{\varphi} &= (\Pi_G\,e^{a\phi})' - \hat{\epsilon}\left(\Pi_{EM} + \frac{m}{2}A_{\varphi}\right)A_{\varphi}' \\
\mathcal{G} &= \hat{\epsilon}\left(\Pi_{EM} - \frac{m}{2}A_{\varphi}\right)'\ .
\end{aligned} \tag{2.16}$$

‡ This metric parameterization is not unique but it accounts for the most generic parameterization for a stationary radial symmetric $2+1D$ metric

$\sqrt{-g} = h\,f$ and the induced $2D$ metric is simply $h_{ij} = \mathrm{diag}(1, h^2)$. The prime $(')$ means the usual derivation (∂_r) with respect to r. We note that $\hat{\epsilon}$ in the gauss constraint is optional once it is a constraint of the gauge sectors only.

For the rotating radially symmetric configurations considered in this work (see subsections 2.1 and 2.2) the only non vanishing gravitational canonical momenta conjugate to h_{ij} is $\pi_G^{r\varphi}$ (conjugate to $h_{r\varphi}$) such that

$$\Pi_G = \mathrm{Tr}(\pi_G) = (\pi_G)^r{}_\varphi = -\frac{h^3 A'}{f} \tag{2.17}$$

and the only non vanishing gauge canonical momenta conjugate to A_i is $\pi^r_{EM} = \delta S/\delta(\partial_0 A_r)$ (conjugate to A_r) such that

$$\Pi_{EM} = \hat{\epsilon}\left(\mathcal{E} - \frac{m}{2}\,A_\varphi\right) \tag{2.18}$$

The contravariant *Electric* and *Magnetic* densities are defined as [37]

$$\begin{aligned} \mathcal{E} &= h\,f\,e^{c\phi}F^{0r} = \hat{\epsilon}\left(\Pi_{EM} + \frac{m}{2}\,A_\varphi\right) \\ \mathcal{B} &= h\,f\,e^{c\phi}\epsilon^{r\varphi}F_{r\varphi} = h\,f\,e^{c\phi}\,A'_\varphi \end{aligned} \tag{2.19}$$

For completeness we also note that the contravariant current densities are defined as

$$\mathcal{J}^\mu = h\,f\,e^{c\phi}J^\mu \tag{2.20}$$

There is a couple of important points that should be stressed. Generally, due to the rotation, magnetic configurations generate a magnetic field and magnetic configurations generate an electric field. However we will solve our equations in the Cartan frame such that for given fields E and B in the Cartan frame we obtain $\mathcal{E} = h\,e^{c\phi}E$ and $\mathcal{B} = h^2\,f\,e^{c\phi}B$. Therefore we don't actually have mixing between electric and magnetic fields (see appendix Appendix A). Ww also note that in $3D$ the magnetic field is a scalar that corresponds in $4D$ to the z-component of the magnetic field, this means the magnetic field perpendicular to the $2D$ spatial coordinates. In our configurations it is null.

The generic gauge canonical momenta are $\pi^i_{EM} = \hat{\epsilon}(hf\,e^{c\phi}F^{0i} - m\epsilon^{ij}A_j/2)$ and therefore π^φ_{EM} is not generally null. However we are only studying configurations in which $F_{t\varphi} = \partial_t A_\varphi - \partial_\varphi A_t = 0$ (see discussion on subsection 2.2) such that $\pi^\varphi_{EM} = -m\,\hat{\epsilon}\,A_r$. Since we are considering only rotating radial symmetric configurations we consider that all the gauge fields are radial functions, furthermore we still have a radial gauge freedom, this means that a gauge transformation $\Lambda(r)$ depending on the radius only has the effect $A_r \to A_r + \Lambda'(r)$ and does not change any of the physical quantities. Therefore we can without lost of generality gauge fix $\pi^\varphi_{EM} = A_r = 0$.

As a final remark note that in the pure Maxwell theory $(m = 0)$ the canonical momentum is proportional to the Electric density $\Pi_{\mathrm{Maxwell}} = \hat{\epsilon}\mathcal{E}$ such that this density is itself a canonical variable, with the Chern-Simons term this is no longer true.

Varying both the action S and the boundary action $A_{\mathcal{B}}$ with respect to the canonical dynamical variables $(h, \Pi_G, \phi, \Pi_{EM}, A_\varphi)$ one obtains a boundary variation [38]

$$\delta S_{\mathcal{B}} = -2\pi\Delta t(-f\delta M + A_0\delta Q + A\delta J) \tag{2.21}$$

where M, Q and J are the Mass, Charge and Angular Momentum of the configuration and $\mathcal{B}$ stands for the one-dimensional spatial boundary of the spatial manifold. Their variation is

$$\begin{aligned}
\delta M &= 2\,\delta\left(h\,e^{a\phi}\right)' + 4\lambda h\phi' e^{a\phi}\,\delta\phi + 2\hat{\epsilon} h e^{c\phi} A'_\varphi\,\delta A_\varphi\Big|_{\mathcal{B}} \\
\delta Q &= 2\hat{\epsilon}\,\delta\left(\Pi_{EM} - \frac{m}{2}\,A_\varphi\right)\Big|_{\mathcal{B}} \\
\delta J &= 2\,\delta\left(\Pi_G e^{a\phi}\right) - 2\hat{\epsilon}\left(\Pi_{EM} + \frac{m}{2}A_\varphi\right)\,\delta A_\varphi\Big|_{\mathcal{B}}
\end{aligned} \tag{2.22}$$

In order to exist well defined classical minimum it is necessary that these variations vanish. We need either to add a boundary action that cancels these variations or to demand them (the variations) to vanish at the boundary. The later is usually a very strong condition and accounts for having expressions for M, Q and J to be constants (meaning r independent). In the absence of external currents the charge Q is necessarily a constant since the Gauss' law is expressed as a total derivative. Accounting with the charge expression, the angular momentum J is also expressed as a total derivative and is therefore a constant as well. For the case of the mass M this is no longer true and we need to add a suitable boundary action. In the presence of external currents neither Q nor J are generally constants since the Gauss' law includes the external charge and is no longer a total derivative, here we are not addressing this case.

Considering the above procedure we obtain

$$\begin{aligned}
M &= 2(h\,e^{a\phi})' + 4\lambda h\phi\phi' e^{a\phi} + 2\hat{\epsilon} h e^{c\phi} A_\varphi A'_\varphi\Big|_{r\to 0}^{r\to\infty} \\
Q &= 2\hat{\epsilon}\left(\Pi_{EM} - \frac{m}{2}\,A_\varphi\right)\Big|_{r\to 0}^{r\to\infty} \\
J &= 2\Pi_G e^{a\phi} - 2\hat{\epsilon}\left(Q + \frac{m}{2}\,A_\varphi\right)A_\varphi\Big|_{r\to 0}^{r\to\infty}
\end{aligned} \tag{2.23}$$

where we used the fact that once the charge constraint in equation (2.22) is taken care, the charge variation vanishes $\delta Q = 0$, and used the expression for the charge to replace $\Pi_{EM} = Q + m\,A_\varphi/2$ in the second term of the equation for the angular momentum variation in order to get a variation of A_φ only. We are considering two disconnected boundaries, the spatial infinite $r=\infty$ and the singularity at the origin $r=0$. We note that these two boundaries have opposite orientations, such that their contributions add up.

As for the mass expression we have to be careful with what fields are fixed and what fields vary upon a functional variation. The correct expression should be

$$M = 2(h\,e^{a\phi})' + 4\lambda h\phi\hat{\phi}' e^{a\hat{\phi}} + 2\hat{\epsilon} h e^{c\hat{\phi}} A_\varphi \hat{A}'_\varphi\Big|_{r\to 0}^{r\to\infty} \tag{2.24}$$

where the hatted fields are fixed at the two boundaries ($r\to 0$ and $r\to\infty$), i.e. upon a functional variation of the mass we obtain the correct expression (2.22).

2.4. Geodesics and Horizons

To compute the geodesics we use the variational principle presented in [39], so we consider the constant functional

$$K = g_{\mu\nu}\frac{x^\mu}{ds}\frac{x^\nu}{ds} = \kappa = \begin{cases} \;\;\,0 & \text{for lightlike (null) geodesics} \\ -1 & \text{for timelike geodesics} \\ +1 & \text{for spacelike geodesics} \end{cases} \tag{2.25}$$

where the derivatives are with respect to a affine parameter s. We minimize K solving the Euler-Lagrange equations $\frac{\delta K}{\delta x^\mu} - \frac{d}{ds}\left(\frac{\delta K}{\delta \dot{x}^\mu}\right) = 0$. Since our solutions are both cylindrically symmetric and stationary (only depend on r, the radial coordinate) we have that the equations for $\mu = t, \varphi$ lead respectively to the first integrals of motion

$$\begin{cases} g_{00}\,\dfrac{dt}{ds} + g_{02}\,\dfrac{d\varphi}{ds} &= E \\[2ex] g_{22}\,\dfrac{d\varphi}{ds} + g_{02}\,\dfrac{dt}{ds} &= L \end{cases} \Rightarrow \begin{cases} \dfrac{dt}{ds} &= \dfrac{E\,g_{22} - L\,g_{02}}{g} \\[2ex] \dfrac{d\varphi}{ds} &= \dfrac{L\,g_{00} - E\,g_{02}}{g} \end{cases} \tag{2.26}$$

with $2E = p_t = \frac{\delta K}{\delta \dot{t}}$ and $2L = p_\varphi = \frac{\delta K}{\delta \dot{\varphi}}$ being constants of motion, the energy and angular momentum respectively (here we rescaled them by a factor of 2 in order to simplify the expressions). Using the two equations (2.26) in (2.25) we obtain an expression for dr/ds

$$\left(\frac{dr}{ds}\right)^2 = k - \frac{L^2\,g_{00} - 2E\,L\,g_{02} + E^2\,g_{22}}{g} \tag{2.27}$$

being g the determinant of the metric $g = g_{00}g_{22} - g_{02}^2$.

Since we are looking for stationary polar symmetric solutions $\dot{t}$ and $\dot{\varphi}$ can be expressed in terms of the radial variable r only, $(d/ds)/(dr/ds) = d/dr$. From the equations for t and φ (2.26) we obtain the differential equations

$$\begin{aligned} t'(r) &= \pm\frac{E\,g_{22} - L\,g_{02}}{\sqrt{g\,(g\kappa - L^2\,g_{00} + 2E\,L\,g_{02} - E^2\,g_{22})}} \\ \varphi'(r) &= \pm\frac{L\,g_{00} - E\,g_{02}}{\sqrt{g\,(g\kappa - L^2\,g_{00} + 2E\,L\,g_{02} - E^2\,g_{22})}} \end{aligned} \tag{2.28}$$

Solving these equation one obtains the t and φ dependence on r. We can also compute the radial velocity $\dot{r} = (dr/ds)/(dt/ds)$ and angular velocity $\dot{\varphi} = (d\varphi/ds)/(dt/ds)$

$$\begin{aligned} \dot{r}(r) &= \pm\frac{\sqrt{g\,(g\kappa - L^2\,g_{00} + 2E\,L\,g_{02} - E^2\,g_{22})}}{E\,g_{22} - L\,g_{02}} \\ \dot{\varphi}(r) &= \frac{L\,g_{00} - E\,g_{02}}{E\,g_{22} - L\,g_{02}} \end{aligned} \tag{2.29}$$

We note that these solutions are for an external observer (at rest far away from the singularity). Then the first equation is particular useful, when $\dot{r} = 0$ we are in the presence either of a turning point on the trajectory, or of a horizon (in which case the geodesics at the rest frame of the travelling observer hits the singularity). We also note that at the singularity, if $\dot{r}$ is null the singularity is not naked, meaning that an external

observer sees the particle stopping when arriving to the singularity. While if $\dot{r}$ has some positive value at the singularity we have a naked singularity since an external observer can actually see it without reaching it. We are using this results to inquire if we have an horizon or not.

3. Electric Solutions

Here we will look for pure Electric solution without external currents, hence we set $B = B_*$, being B the magnetic field in the Cartan frame and B_* the magnetic field in the original frame. We will be working in the Cartan frame and at the end of each subsection we will summarize our results in the original frame. The equations of motion in the Cartan frame are computed in appendix Appendix A and are equivalent to the equations of motion as presented in subsection 2.1.

From the first Maxwell Equation (A.24) we obtain that

$$\gamma = m\, e^{-c\phi} \tag{3.1}$$

Using (3.1) in (A.26) one gets that $\beta = c\phi'/2$ and from the definition of β (see (A.21) in appendix) we get the solution for h

$$h = c_h\, e^{\frac{c}{2}\phi} \tag{3.2}$$

where c_h is a free integration constant.

Now we get from the second Maxwell Equation (A.25) that

$$E = \chi e^{-\frac{3}{2}c\phi} \tag{3.3}$$

where χ is an integration constant. Note that without loss of generality we included c_h in the definition of this constant. There is a very important conclusion to take from this last equation, trivial solutions for the scalar field (ϕ = constant) holds in the Cartan frame a uniform (constant) electric field E in all space, this conclusion was firstly obtained in [2]. Although for completeness we address trivial solutions we will first address non-trivial solutions for the scalar field which is the main objective of this work.

3.1. Non-Trivial Solutions for the Scalar Field

We will now address the full equations considering the generic equations. The three Einstein (A.27-A.29) and scalar field equations (A.30) read now

$$(a+\frac{c}{2})\phi'' + (a^2 - \frac{\lambda}{2} + \frac{c^2}{4})(\phi')^2 + \frac{m^2}{4}e^{-2c\phi} + \frac{\Lambda}{2}e^{(b-a)\phi} = -\,\hat{\epsilon}\chi^2 e^{(-a-2c)\phi} \tag{3.4}$$

$$a\phi'' + (a^2 - \frac{\lambda}{2})(\phi')^2 + \alpha^2 + \alpha' - \frac{3m^2}{4}e^{-2c\phi} + \frac{\Lambda}{2}e^{(b-a)\phi} = \hat{\epsilon}\chi^2 e^{(-a-2c)\phi} \tag{3.5}$$

$$\frac{\lambda}{2}(\phi')^2 + \frac{c}{2}\alpha\phi' + \frac{m^2}{4}e^{-2c\phi} + \frac{\Lambda}{2}e^{(b-a)\phi} = -\,\hat{\epsilon}\chi^2 e^{(-a-2c)\phi} \tag{3.6}$$

$$(4a^2-\lambda)\phi''+a(4a^2-2\lambda)(\phi')^2+(3a-b)\Lambda e^{(b-a)\phi} \quad = -\hat{\epsilon}(a+c)\chi^2 e^{(-a-2c)\phi} \tag{3.7}$$

The main problem to solve these equations is to make them compatible with each other in order to give a non trivial solution. For $a=b=c$, for $a=0$ (any b and c), for $b=0$ (any a and c) and $c=0$ (any a and b) these equations hold that the scalar field has only trivial solutions, i.e. it must be a constant. Trivial solutions will be addressed in the next subsections. For the particular cases $c=0$ with $a=b$ and $a=b=-2c$ solutions do exist but hold that the scalar field is purely imaginary.

The better way to properly understood the structure of the equations is the following. The third equation (3.6) can be algebraically solved in α which solution is then plugged into the second equation (3.5). Then to obtain a solution for the ϕ we can make a linear combination of the remaining three equations obtaining a simpler equation. The main problem then is to ensure that the solution is compatible with the original equations (or equivalently with different linear combinations of the original equations). This procedure gives very few choices for non-trivial solutions.

We only found non-trivial solutions for the case

$$\begin{aligned} a &= 0 \\ c &= -\frac{b}{2} \\ \lambda &\neq \frac{b^2}{8} \end{aligned} \tag{3.8}$$

For $b^2=8\lambda$ does not exist a non-trivial solution either. We note that for the choice of equation (3.8) we are not working with dilaton Einstein theory. Our action is more similar to what is commonly know as a gravitational scalar field [27, 28, 30, 31] and the cosmological constant term resembles a Dilaton potential [50] §

Given this ansatz we combine (3.4) with (3.7) obtaining

$$\phi'=\pm\sqrt{c_1}e^{b\,\phi} \tag{3.9}$$

such that

$$\phi=-\frac{2}{b}\ln(c_\phi(r-r_0)) \tag{3.10}$$

Here

$$c_\phi=\frac{|b|}{2}\sqrt{c_1} \qquad c_1=-2\frac{b^2(\hat{\epsilon}\chi^2+2\Lambda)+2\lambda(4\hat{\epsilon}\chi^2+2\Lambda+m^2)}{\lambda(b^2-8\lambda)} \tag{3.11}$$

and without loss of generality we set the integration constant $r_0=0$ since it represents only a shift in the radial coordinate and all the solutions depend on the ϕ exponentials. Note that the choice of sign in (3.9) depends on the sign of b such that in (3.10) the argument of the logarithm is positive. Also we have to ensure that c_1 is positive defined. Before doing so we use the ϕ solution (3.10) in (3.4). In order the equation to be solved we have to impose

$$\chi^2=-\hat{\epsilon}\frac{2\Lambda(b^2+12\lambda)+4\lambda m^2}{b^2+24\lambda} \tag{3.12}$$

§ Thanks to Dmitri Gal'tsov for this remark.

Now c_1 becomes

$$c_1 = 4\,\frac{m^2 - 6\Lambda}{b^2 + 24\lambda} \tag{3.13}$$

From (3.6) and the definition $\gamma = A'h/f$ (A.21) we get that

$$\alpha = -\left(16\frac{\lambda}{b^2} + 1\right)\frac{1}{2\,r} \tag{3.14}$$

Therefore from the definition of $\alpha = f'/f$ (see (A.21) in the appendix) we obtain the solution for f

$$f = c_f\, r^{-\frac{8\lambda}{b^2} - \frac{1}{2}} \tag{3.15}$$

from (3.2) we obtain the solution for h

$$h = c_h\,\sqrt{r} \tag{3.16}$$

and from (3.1) we get the solution for A

$$A = c_A r^{-\frac{8\lambda}{b^2} - 1} + c_{A\infty} \tag{3.17}$$

where

$$c_A = \frac{m\,c_f}{c_h\left(-\frac{8\lambda}{b^2} - 1\right)}\sqrt{\frac{1 + \frac{24\lambda}{b^2}}{m^2 - 6\Lambda}} \tag{3.18}$$

c_f, c_h and $c_{A\infty}$ are free constants.

Replacing these solutions in the remaining equation (3.5) and demanding it to be obeyed we get that

$$\lambda_\pm = \frac{b^2}{8}\,\frac{3\Lambda \mp \sqrt{\Lambda(2m^2 - 3\Lambda)}}{m^2 - 6\Lambda} \tag{3.19}$$

We have to ensure that all these relations are possible and that do not correspond to trivial solutions, in particular that $\chi^2 > 0$ and $C_1 > 0$. Therefore for each $\hat{\epsilon} = \pm 1$ we have to choose the solution $\lambda_{\hat{\epsilon}}$ getting

$$\begin{aligned} \chi^2 &= \frac{1}{2}\left[-\hat{\epsilon}\Lambda + \sqrt{\Lambda(2m^2 - 3\Lambda)}\right] \\ C_1 &= \frac{4}{b^2}\left[3\Lambda + m^2 + 3\hat{\epsilon}\sqrt{\Lambda(2m^2 - 3\Lambda)}\right] \end{aligned} \tag{3.20}$$

Demanding positiveness of these expressions hold, independently of $\hat{\epsilon}$ the same constraint on the cosmological constant Λ and topological mass m

$$0 < \Lambda < \frac{m^2}{2} \tag{3.21}$$

For the particular value of $\Lambda = m^2/6$ some of the expressions previously computed are not well defined. It is necessary to rederive the solution using the same method. For $\hat{\epsilon} = +1$ we obtain

$$\Lambda = m^2/6 \quad C_1 = \frac{12m^2}{b^2} \quad \chi^2 = \frac{m^2}{6} \quad \lambda = -\frac{b^2}{24} \quad C_A = \frac{\sqrt{3}c_f}{c_h}\;. \tag{3.22}$$

All the other solutions remain the same up to replacement of the above constants. For $\hat{\epsilon}=-1$ there are no allowed solutions at $\Lambda = m^2/6$.

For convenience we define the parameter p which depends only on the ratio Λ/m^2

$$p = -8\frac{\lambda}{b^2} = \frac{-3x + \hat{\epsilon}\sqrt{x(2-3x)}}{1-6x} \qquad x = \frac{\Lambda}{m^2} \tag{3.23}$$

For clarity we summarize and rewrite the solutions computed above in the original frame,

$$\begin{aligned}
\phi &= -\frac{2}{b}\ln(C_\phi\, r) \\
h &= C_h\,\sqrt{r} \\
f &= C_f\, r^{p-\frac{1}{2}} \\
A &= C_A\, r^{p-1} + \theta \\
E_* &= C_E\, r^{p-2} \\
A_0 &= \frac{C_E}{p-1}\, r^{p-1}
\end{aligned} \tag{3.24}$$

where for convenience we rename the variables and integration constants. C_h, C_f, b and θ which are free parameters while the remaining variables are

$$\begin{aligned}
p &= -\frac{3\Lambda - \hat{\epsilon}\sqrt{\Lambda(2m^2-3\Lambda)}}{m^2-6\Lambda} \\
\lambda &= -\frac{8}{b^2}p \\
C_\phi &= \sqrt{\frac{m^2-6\Lambda}{1-3p}} \\
C_A &= \frac{m\,C_f}{C_h\,(p-1)}\sqrt{\frac{1-3p}{m^2-6\Lambda}} \\
C_{E(\pm)} &= \mp\frac{C_f}{\sqrt{2}}\sqrt{4\Lambda - p(m^2+6\Lambda)}\left(\frac{1-3p}{(m^2-6\Lambda)^3}\right)^{\frac{1}{4}}
\end{aligned} \tag{3.25}$$

Here $\theta = C_{A_\infty}$ in (3.17). For $\hat{\epsilon}=+1$ and the particular case $\Lambda = m^2/6$ corresponding to $p = 1/3 \Leftrightarrow \lambda = -b^2/24$ we have $C_A = \sqrt{3}C_f/(2C_h)$. The values of the remaining constants are well defined, $C_\phi = \sqrt{3}\,m$ and $C_{E(\pm)} = \mp 3\,C_f\, m^{5/2}/\sqrt{2}$. For the values $p=0$ $(\Lambda=0)$ and $p=1/2$ $(\Lambda = m^2/2)$ we obtain $C_E = 0$ and therefore the solutions

presented here do not allow charged configurations for these particular limit values. In these cases $C_\phi \sim m$. For $\hat{\epsilon} = -1$ the particular case $\Lambda = m^2/6$ has no real solutions.

We have the bound in the cosmological constant

$$0 < \Lambda < \frac{m^2}{2} \tag{3.26}$$

such that p is in the range

$$\begin{cases} p \;\in\; \left]0, \dfrac{1}{2}\right[& \hat{\epsilon} = +1 \\ \\ p \;\in\;]-\infty, 0[\, \cup\,]1, +\infty[& \hat{\epsilon} = -1 \end{cases} \tag{3.27}$$

where for both cases $p = 0$ corresponds to $\Lambda = 0$ and for $\hat{\epsilon} = +1$ we have $p = 1/2$ corresponding to $x = \Lambda/m^2 = 1/2$ while for $\hat{\epsilon} = -1$ we have $p = 1$ corresponding to $x = \Lambda/m^2 = 1/2$. For $\hat{\epsilon} = +1$ we have that $p = 1/3$ corresponds to $x = \Lambda/m^2 = 1/6$ while for $\hat{\epsilon} = -1$ we have that $\lim_{x\to(1/6)^\pm} p = \mp\infty$. For $\hat{\epsilon} = -1$, $p \in]-\infty, 0[$ corresponds to $x = \Lambda/m^2 \in]0, 1/6[$ and $p \in]1, \infty[$ corresponds to $x = \Lambda/m^2 \in]1/6, 1/2[$.

3.2. Trivial Scalar Field Solutions: $\phi = 0$

It remains to analyse the case of $\phi = 0$. This case corresponds to not considering the scalar field at all and has been first addressed by Kogan [2], however in the original work a cosmological constant have not been considered (it has in [3] but without solving the equations of motion), for this reason we also discuss it here.

Considering the above solutions for γ (3.1), h (3.2) and E (3.3) the remaining three Einstein (A.27-A.29) reduce only to two independent equations

$$\frac{m^2}{4} + \frac{\Lambda}{2} = -\,\hat{\epsilon}\chi^2 \tag{3.28}$$

$$\alpha^2 + \alpha' - \frac{3m^2}{4} + \frac{\Lambda}{2} = \hat{\epsilon}\chi^2 \tag{3.29}$$

while the scalar field equation (A.30) is already obeyed. Solving the first equation for χ^2 we get

$$\chi^2 = -\hat{\epsilon}\left(\frac{m^2}{4} + \frac{\Lambda}{2}\right) \tag{3.30}$$

and demanding the right hand side to be positive definite we obtain the constraint

$$\begin{cases} \Lambda \;<\; -\frac{m^2}{2} & \hat{\epsilon} = +1 \\ \\ \Lambda \;>\; -\frac{m^2}{2} & \hat{\epsilon} = -1 \end{cases} \tag{3.31}$$

As in the previous subsection in order to exist electric solutions the cosmological constant is constraint to be negative for $\hat{\epsilon} = +1$ and can be both negative in the range $]-m^2/2, 0[$ or positive for $\hat{\epsilon} = -1$. We also note that from (3.30) the equality $\Lambda = m^2/2$ holds that $\chi = E = 0$, therefore not allowing electric configurations. For this reason we don't consider the case $\Lambda = -m^2/2$.

From equation (3.29) and the definition of α (see (A.21) in appendix) we get the solution for f

$$f = c_f \cosh\left(\sqrt{k}\,(r - r_0)\right) \tag{3.32}$$

where c_f and r_0 are integration constants and

$$k = \frac{m^2}{2} - \Lambda\ . \tag{3.33}$$

From (3.2) we have that

$$h = c_h \tag{3.34}$$

and from (3.1) and the definition for γ (A.21) we obtain the solution for A

$$A = \frac{m\, c_f \sinh\left(\sqrt{k}\,(r - r_0)\right)}{c_h \sqrt{k}} + c_{A_0} \tag{3.35}$$

where c_{A_0} is an integration constant that corresponds to the value of A at $r = r_0$. Again we can set $r_0 = 0$ since it represents a shift in the radial coordinate.

For clarity we summarize and rewrite the solutions just obtained in the original frame

$$\begin{aligned} h &= C_h \\ f &= C_f \cosh(K\, r) \\ A &= C_A \sinh(K\, r) + \theta \\ E_* &= C_E \cosh(K\, r) \\ A_0 &= \frac{C_E}{K} \sinh(K\, r) \end{aligned} \tag{3.36}$$

where C_h and C_f are free constants and the remaining constants are defined as

$$\begin{aligned} K &= \sqrt{\frac{m^2}{2} - \Lambda} \\ C_A &= \frac{m\, C_f}{C_h\, K} \\ C_{E(\pm)} &= \pm\frac{C_f}{2}\sqrt{\left|\frac{m^2}{2} + \Lambda\right|} \end{aligned} \tag{3.37}$$

The cosmological constant is constraint and accordingly K is real for $\hat{\epsilon} = +1$

$$\hat{\epsilon} = +1\ :\quad \begin{cases} \Lambda < -\dfrac{m^2}{2} \\ K \in \,]m^2, +\infty[\end{cases} \tag{3.38}$$

but can be both real and imaginary for $\hat{\epsilon} = -1$

$$\hat{\epsilon} = -1 \ : \quad \left\{ \begin{array}{lcl} \Lambda & \in & \left]-\frac{m^2}{2}, 0\right[\\ K & \in &]0, m^2[\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{lcl} \Lambda & \in &]0. +\infty[\\ K & \in &]0, +\infty[\ i \end{array} \right. \tag{3.39}$$

where the last interval for K is imaginary. In this last case we obtain periodic solutions in r with period $2\pi/|K|$.

As a final remark we note that the contravariant electric density as defined in (2.19) is a constant

$$\mathcal{E} = -\frac{C_E\, C_h}{C_f} \tag{3.40}$$

as expected from the solution for E in the Cartan frame.

We already analyse the case for a null scalar field, but a constant scalar field is also an allowed trivial solution. For such solutions we obtain the same solutions up to the redefinition of the parameters

$$\tilde{\Lambda} = \Lambda\, e^{(b-a)\phi} \qquad \tilde{m} = m\, e^{-c\phi} \qquad \tilde{\chi} = \chi\, e^{-\frac{a}{2}-c} \tag{3.41}$$

with $\phi =$ constant.

4. Singularities, Geodesics and Horizons

4.1. Non-Trivial Solutions

The contraction of the Ricci tensor is

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} &= \frac{1}{4C_f^4 r^4}[C_f^4(3+2p(-8+p(17+2p(-7+2p))))- \\ &\quad 2(C_A C_f C_h(p-1))^2(3+4p(p-2))r + 3(C_A C_h(p-1))^4] \end{aligned} \tag{4.1}$$

which shows that there is a curvature singularity at $r = 0$. The curvature is

$$R = \frac{h^3 A'^2 - 4f(hf'' + f'h' + fh'')}{2f^2 h} = \frac{m^2 p(3-4p) + 6\Lambda(4p^2 - 6p + 1)}{2(m^2 - 6\Lambda)}\,\frac{1}{r^2}\ . \tag{4.2}$$

For $\hat{\epsilon} = +1$ we have always positive curvature while for $\hat{\epsilon} = -1$ we can have both negative and positive curvatures

$$\begin{array}{lclll} r^2 R & \in & \left]0, \frac{5}{8}\right[& \text{for} \quad \hat{\epsilon} = +1 \ \text{ and } \ x \in \left]0, \frac{1}{2}\right[\\ r^2 R & \in &]-\infty, 0[& \text{for} \quad \hat{\epsilon} = -1 \ \text{ and } \ x \in \left]0, \frac{9}{38}\right[/ \left\{\frac{1}{6}\right\} \\ r^2 R & = & 0 & \text{for} \quad \hat{\epsilon} = -1 \ \text{ and } \ x = \frac{9}{38} \\ r^2 R & \in & \left]0, \frac{9}{8}\right] & \text{for} \quad \hat{\epsilon} = -1 \ \text{ and } \ x \in \left]\frac{9}{38}, \frac{1}{2}\right[\end{array} \tag{4.3}$$

For the limiting cases $\Lambda \to 0$ (corresponding to $x \to 0$) we have $R \to 0$ for both $\hat{\epsilon} = \pm 1$ and for $\Lambda \to m^2/2$ (corresponding to $x \to 1/2$) we have $R \to 5/(8r^2)$ for $\hat{\epsilon} = +1$ and

$R \to 1/r^2$ for $\hat{\epsilon} = -1$. The Maximum value of the curvature for the case $\hat{\epsilon} = -1$ is $R = 9/(8r^2)$ corresponding to $x = 9/26$. We note that as already explained in the last section $x = 0$ and $x = 1/2$ are not allowed solutions and, for $\hat{\epsilon} = -1$, $x = 1/6$ is neither an allowed solution.

For both cases $\hat{\epsilon} = \pm 1$ the curvature is asymptotically flat ($\lim_{r\to\infty} R = 0$), therefore our spaces are asymptotically flat.

In order to find if there is or not an horizon it is enough to consider a photon travelling in the radial direction. So we can solve equations (2.28) with $L = 0$ and $\kappa = 0$ obtaining

$$\begin{aligned} t(r) &= t_0 \pm \frac{2}{|C_f|(2p-3)} r^{3/2-p} \\ \varphi(r) &= \varphi_0 \pm \frac{2\left((2p-3)r^{p-1} - \theta\right)}{|C_f|(2p-3)} r^{3/2-p} \end{aligned} \tag{4.4}$$

For $\hat{\epsilon} = +1$ we have that $p \in]0, 1/2[$, so these solutions are regular for all r and we conclude that there is no horizon. From regularity at the singularity $r = 0$ we are in the presence of a naked singularity, for an external observer the photon will hit the singularity in a finite time. For $\hat{\epsilon} = -1$ we can have an horizon at $r = 0$ as long as $p > 3/2$ ($p = 3/2 \Leftrightarrow x = \Lambda/m^2 = 9/26$). This will happen for

$$x = \frac{\Lambda}{m^2} \in \left]\frac{1}{6}, \frac{9}{26}\right[\ . \tag{4.5}$$

Then in this range we will have a dressed singularity, for an external observer the infalling particle will take an infinite amount of time to reach the singularity. For all other values of p we have a naked singularity. We note that from (4.4) for $p = 3/2$ the geodesics are a fixed point on time and there are no horizons.

In order to understand the meaning of our singularity in terms of the angular variable let us now compute the angle deficit of our space, or equivalently the maximum value for the angular variable φ. The metric reads

$$ds^2 = -r^{2p-1}\,dt^2 + dr^2 + C_h^2 r\,\left(d\varphi + A\,dt\right)^2 \ . \tag{4.6}$$

Let us remember from the discussion in section 2 that the $2D$ induced metric is $h_{ij} = \mathrm{diag}(1, h^2) = \mathrm{diag}(1, C_h^2\, r)$. Now let us make a transformation of coordinates $r \to \tilde{r}$ such that the measure of the induced metric is the usual one, i.e $\sqrt{|h_{ij}|} = \tilde{r}^2$. This accounts for a observer at rest in relation to space-time (hence rotating with space). The transformation of the radial coordinate is

$$r = \left(\frac{3}{4C_h}\right)^{\frac{2}{3}} \tilde{r}^{\frac{4}{3}} \quad \Rightarrow \quad \left\{ \begin{aligned} f &= \left(\frac{3\tilde{r}^2}{4C_h}\right)^{\frac{2}{3}\left(p-\frac{1}{2}\right)} \\ h_{rr} &= \left(\frac{4\tilde{r}}{3C_h^2}\right)^{\frac{2}{3}} \\ h_{\varphi\varphi} &= \left(\frac{3C_h^2\tilde{r}^2}{4}\right)^{\frac{2}{3}} \end{aligned} \right. \tag{4.7}$$

The maximum angle is computed as

$$\varphi_{\max} = \frac{2\pi}{\sqrt{-g}}\sqrt{\frac{h_{\varphi\varphi}}{h_{rr}}} = \frac{2\pi}{f\,h_{rr}} \ . \tag{4.8}$$

In order to obtain the background geometry we take the limit $p \to 0$ (equivalent to $\Lambda \to 0$). We will discuss this limit properly in the next section when computing the mass, charge and angular momentum, for now let us just take it as granted, then the respective maximum angle is

$$\varphi_{\max} = 2\pi\,\frac{3|C_h|}{4} \ . \tag{4.9}$$

Imposing it to be as usal 2π we obtain the value for C_h

$$|C_h| = \frac{4}{3} \ . \tag{4.10}$$

So we have a rotating background without any angle deficit. For generic p we obtain

$$\varphi_{\max} = 2\pi\left(\frac{3\tilde{r}}{4}\right)^{-\frac{4}{3}p} \tag{4.11}$$

such that for $\tilde{r} = 4/3$ we have $\varphi_{\max} = 2\pi$ always. In the limit $\tilde{r} \to 0$ we obtain that for $p > 0$, $\varphi_{\max} \to \infty$ and for $p < 0$, $\varphi_{\max} \to 0$. While in the limit $\tilde{r} \to \infty$ we obtain that for $p > 0$, $\varphi_{\max} \to 0$ and for $p < 0$, $\varphi_{\max} \to \infty$. So we conclude that only for $p < 0$ the singularity is a conical singularity (in the usual sense that we get an angular deficit), while for $p > 0$ what we obtain as $\tilde{r} \to 0$ is not a deficit, but instead a decompactification of the angular variable.

Then we have the following cases

$$\begin{array}{llcll}
\hat{\epsilon} = +1 & x \in \left]0, \frac{1}{2}\right[& \Rightarrow & p \in \left]0, \frac{1}{2}\right[: & \text{decompactification singularity} \\
\hat{\epsilon} = -1 & x \in \left]0, \frac{1}{6}\right[& \Rightarrow & p \in \left]0, +\infty\right[: & \text{decompactification singularity} \\
\hat{\epsilon} = -1 & x \in \left]\frac{1}{6}, \frac{1}{2}\right[& \Rightarrow & p \in \left]-\infty, -1\right[: & \text{conical singularity}
\end{array} \tag{4.12}$$

4.2. Trivial Scalar Field Solutions

The contraction of the Ricci tensor is a constant

$$R_{\mu\nu}R^{\mu\nu} = \frac{K^4}{4C_f^4}\left(8C_f^4 - 8C_A^2C_f^2C_h^2 + 3C_A^4C_h^4\right) \tag{4.13}$$

which indicates that the space-time has no singularities. Specifically the curvature is

$$R = -\frac{K^2}{2}\left(4 - \frac{C_A^2C_h^2}{C_f^2}\right) = -\frac{m^2}{2} + 2\Lambda \tag{4.14}$$

and can have either positive or negative values. Taking in account the bounds for the cosmological constant (3.31) we obtain that

$$\begin{aligned} R &< 0 \qquad \text{for} \quad \hat{\epsilon} = +1 \quad \text{and} \quad \Lambda < -\frac{m^2}{2} \\ R &< 0 \qquad \text{for} \quad \hat{\epsilon} = -1 \quad \text{and} \quad \Lambda \in \left]-\frac{m^2}{2}, \frac{m^2}{4}\right[\\ R &= 0 \qquad \text{for} \quad \hat{\epsilon} = -1 \quad \text{and} \quad \Lambda = \frac{m^2}{4} \\ R &> 0 \qquad \text{for} \quad \hat{\epsilon} = -1 \quad \text{and} \quad \Lambda \in \left]\frac{m^2}{4}, +\infty\right[\ . \end{aligned} \tag{4.15}$$

Therefore we conclude we are in the presence of an extended (non localized) configuration, there is no singularity, hence this solution cannot be considered as a classical particle. We recall that for $\hat{epsilon} = +1$ and $\Lambda \geq -m^2/2$ and for $\hat{epsilon} = -1$ and $\Lambda \leq -m^2/2$ there are no allowed solutions.

5. Mass, Charge and Angular Momentum

In this section we compute the mass, charge and angular momentum.

5.1. Non-Trivial Scalar Field Solutions

As expected the Hamiltonian Constraint $\mathcal{H} = 0$, Momentum Constraint $\mathcal{H}^\varphi = 0$ and Gauss Constraint $\mathcal{G} = 0$ are obeyed, this is actually a way to check that our calculations are correct.

Using (2.23) we obtain that the Mass of the configuration is

$$\begin{aligned} M &= \left.(2h' + 4\,\lambda\,h\,\phi\,\phi')\right|_{r\to\delta_M}^{r\to\infty} = \frac{b^2+16\lambda}{b^2}\,C_h\,\left.\frac{\ln(C_\phi\,r)}{\sqrt{r}}\right|_{r\to\delta_M}^{r\to\infty} = \\ &= -2\,C_h\,p\,\frac{\ln(C_\phi\,\delta_M)}{\sqrt{\delta_M}} \end{aligned} \tag{5.1}$$

We introduced a cut-off $\delta_M \ll 1$ because this quantity has a infrared divergence as we compute the limit of $\delta_M \to 0$.

The charge of this configuration is computed to be

$$Q_e = -\frac{2\,C_h\,C_\phi\,C_E}{C_f} \tag{5.2}$$

The constant C_f can be set to unity by a proper redefinition of time $t \to t/C_f$ and the redefinitions of the remaining constants $C_h \to C_h/C_f$ and $\theta \to \theta/C_f$. So without any loss of generality we set $C_f = 1$. However we must remember that C_E as given in (3.25) has no defined sign and we must demand that the electric field has the correct sign when compared with the charge. From (5.2) we conclude that in order Q_e and C_E

to have the same sign we are left only with the possibility of $C_h < 0$, then $C_h = -4/3$. Then we rewrite the charge as

$$Q_e = \pm\frac{2\sqrt{2}}{3}\frac{\sqrt{m^2 p - 2\Lambda(2-3p)}}{((m^2-6\Lambda)(1-3p))^{\frac{1}{4}}} \tag{5.3}$$

where the $\pm$ accounts for positive and negative charge configurations. C_E must account for this and the sign is set accordingly

$$C_E \sim \text{sign}\,(Q_e) \tag{5.4}$$

As we can see from (5.1) this choice of signal for C_h affects the mass sign, the mass is positive or negative depending on the sign of p. We note that the logarithm in (5.1) is negative and therefore the mass is positive when $p < 0$ and negative when $p > 0$. For $\hat{\epsilon} = +1$ it is always negative, while for $\hat{\epsilon} = -1$ it is negative for $\Lambda \in]0, m^2/6[$ and positive for $\Lambda \in]m^2/6, m^2/2[$

There is also one interesting point concerning the discrete symmetries time-inversion T and P. Inverting time accounts for choosing $C_f = -1$ such that $t \to -t$. The visible direct effects of the transformation $C_f \to -C_f$ for our solutions is to invert the sign of C_A and C_E (assuming we have fixed the $\pm$ of C_E, see (3.25)). Doing so we revert the sign of the charge definition as it depends explicitly on C_f as well, see (5.2)), and although $C_E \to -C_E$, our charge maintains its sign. Then we have two problems, first the charge and the electric field have now the wrong relative sign (we are considering $C_h < 0$ fixed) and secondly the charge is not transforming properly under T (see for instance equation (50) of [40], see also [24]). Therefore we are forced to transform $C_h \to -C_h$ as well obtaining $C_h > 0$. As a consequence C_A does not actually changes sign (because the ratio C_h/C_f does not change), this accounts fot T violation due to the Chern-Simons term. Also we note that by choosing $C_f = -1$ (or transforming $C_f \to -C_f$ and $C_h \to -C_h$) inverts the mass sign. This is actually expected, we recall the reader that classically a positron looks like an electron travelling backwards in time. As for parity P, will account for the transformation $C_h \to -C_h$ which from the above discussion implies as well $C_f \to -C_f$ and we obtain the same effects.

The Angular Momentum of this configuration is

$$J = -\frac{2\,C_h^3\,C_A\,(p-1)}{C_f} - J_0 = \frac{28\,m}{9}\sqrt{\frac{1-3p}{m^2-6\Lambda}} - J_0 \tag{5.5}$$

where J_0 is the background angular momentum and will be computed later. the sign of J does not depend in the particular configuration, but only on the relative sign between the Maxwell term (F^2) and the Chern-Simons term ($A \wedge F$) as explained on subsection 2.2. This means it will change if we consider the transformations $m \to -m$ and vanishes for $m = 0$ (as will be shown it does not vanishes in the limits $m \to 0^{\pm}$, only for $m = 0$). This is clearly also an effect of T and P violation which is expected when a Chern-Simons term is present.

So as we have just seen our solutions violate both T and P as expected when a Chern-Simons term is present. This is explicit on the fact that the signs of C_A and J only depend on the relative sign between the Maxwell and the Chern-Simons term.

We already computed the angle deficit in the last section such that for $C_h = -4/3$ our background has the correct angular variable $\varphi \in [0, 2\pi[$. Here we still have to compute J_0, so we are properly explaining what are the limits of our solutions when we take the Chern-Simons coefficient to zero, $m \to 0$. From the constraint interval we have that it corresponds to $\Lambda \to 0$ (equivalent to $x = \Lambda/m^2 \to 0$ and $p \to 0$). We will analyse this limit from the definitions (3.25). In this limit we obtain from (3.25) that $C_E \to 0$ therefore we have necessarily $Q_e \to 0$, also we obtain $C_\phi \to 0$ and $C_A \to -\,\mathrm{sign}\,(m) C_f/C_h$. We note that for C_A the limits on the right and left $(m^\pm)$ are finite with opposite signs such that for $x = p = 0$ we obtain $C_A = 0$. Nevertheless the asymptotic limit are defined only from the left and from the right such that for the limiting cases $C_A \neq 0$. One obtains from (5.5) that $J \to -2\,\mathrm{sign}\,(m) C_h^2 - J_0$. The first term corresponds to the background angular momentum, therefore we obtain

$$J_0 = -2\,\mathrm{sign}\,(m) C_h^2 \ . \tag{5.6}$$

As already expected its sign depends on the relative sign between the Maxwell and the Chern-Simons term and accounts for parity violation. One obtains by a direct computation that $M \to 0$ and also that the curvature vanishes everywhere, $R \to 0$. Therefore as background for our configurations we obtain a stationary rotating flat space without any angle deficit as already studied in the last section. The background metric is

$$ds^2 = -\frac{1}{r}\,dt^2 + dr^2 + C_h^2 r \left(d\varphi + \left(-\frac{\mathrm{sign}\,(m)}{C_h}\frac{1}{r} + \theta \right) dt \right)^2 \ . \tag{5.7}$$

5.2. Trivial Scalar Field Solutions

We will now compute the charge, mass and angular momentum for the trivial solution (3.36) with $\phi = 0$.

The mass of the configuration is null, the charge is

$$Q_e = -\frac{C_E\,C_h}{C_f} = \pm\frac{C_h}{2}\sqrt{\left|\frac{m^2}{2} + \Lambda\right|} \tag{5.8}$$

and the angular momentum is

$$J = -\frac{C_A\,C_h^3\,K}{C_f} - J_0 = -m\,C_h^2 - J_0 \ . \tag{5.9}$$

We note that again the sign of the angular momentum only depends on the relative sign between the Maxwell and Chern-Simons term.

We can solve (5.8) for C_h obtaining

$$C_h = \frac{2Q_e}{\sqrt{\left|\frac{m^2}{2} + \Lambda\right|}} \tag{5.10}$$

and

$$J = -\frac{4m\,Q_e^2}{\left|\frac{m^2}{2} + \Lambda\right|} - J_0 \tag{5.11}$$

Now the $\pm$ in C_E must be chosen accordingly to the sign of the charge such that we obtain

$$C_E = \frac{\mathrm{sign}\,(Q_e) C_f}{2} \sqrt{\left|\frac{m^2}{2} + \Lambda\right|} \tag{5.12}$$

Again we can redefine $t \to t/C_f$ that corresponds to set $C_f = 1$.

By computing the limit $m \to 0$ we obtain that $C_A \to 0$, therefore both the charge and angular momentum vanish and we obtain the flat space

$$ds^2 = -C_f dt^2 + dr^2 + C_h \left(d\varphi + \theta\, dt\right)^2 \; . \tag{5.13}$$

Using the same procedure we obtain that the angular variable is in the range $\varphi \in [0, 1/r^2[$, so this space has some pathologies.

6. Summary and Discussion of Results

6.1. Summary of Non-Trivial Solutions

We will briefly resume the results obtained in this paper. Although we are repeating some of the equations of the article we think it is necessary in order to assemble and clarify all the results obtained.

We found a electric point particle that can constitute either a naked or dressed singularity, depending on the parameter choices. The results are presented in terms of $x = \Lambda/m^2$, the cosmological constant to topological mass squared (Chern-Simons coefficient squared) ratio and the charge Q_e of the configuration.

The metric, scalar field and gauge field solutions for such configuration are

$$\begin{aligned}
ds^2 &= \left(\frac{16}{9} r \left(C_A\, r^{p-1} + \theta\right)^2 - r^{2p-1}\right) dt^2 + dr^2 \\
&+ \frac{16}{9} r\, d\varphi^2 + \frac{16}{9} r \left(C_A\, r^{p-1} + \theta\right) dt\, d\varphi \\
\phi &= -\frac{2}{b} \ln(|m| \sqrt{\frac{1-6x}{1-3p}} r) \\
A_0 &= \frac{C_E}{p-1} r^{p-1}
\end{aligned}$$

where θ and b are free parameters and all the remaining constants depend only on the cosmological constant to Chern-Simons square coefficient ratio $x = \Lambda/m^2$

$$\begin{aligned}
p &= -\frac{3x - \hat{\epsilon}\sqrt{x(2-3x)}}{1-6x} \\
C_A &= \frac{3\,\mathrm{sign}\,(m)}{4(1-p)} \sqrt{\frac{(1-3p)}{(1-6x)}} \\
C_E &= \frac{\mathrm{sign}\,(Q_e) \sqrt{p + 2x(3p-2)}}{\sqrt{2|m|}} \left(\frac{1-3p}{(1-6x)^3}\right)^{\frac{1}{4}}
\end{aligned}$$

The Brans-Dicke coefficient is determined up to the free parameter b as $\lambda = -8p/b^2$ and the remaining scalar field exponential coefficients are fixed, $a = 0$ and $c = -b/2$.

The charge, angular momentum and mass are

$$\begin{aligned}
Q_e &= \pm\frac{2\sqrt{2}}{3}\frac{\sqrt{p-2x(2-3p)}}{((1-6x)(1-3p))^{\frac{1}{4}}} \\
J &= -\frac{28\,\mathrm{sign}\,(m)}{9}\left(\sqrt{\frac{1-3p}{1-6x}}-1\right) \\
M &= \frac{8p}{3}\frac{\ln(C_\phi\,\delta_M)}{\sqrt{\delta_M}}\frac{\ln\left(|m|\sqrt{\frac{1-6x}{1-3p}}\delta_M\right)}{\sqrt{\delta_M}}
\end{aligned} \tag{6.14}$$

The mass is infrared divergent and we consider a cut-off proportional to the Planck Length, $\delta_M \sim l_p = \sqrt{G}$, being G the Newton gravitational constant in natural units.

The curvature is

$$R = \frac{3\hat{\epsilon}\sqrt{x(2-3x)} + x(-11+48x-12\hat{\epsilon}\sqrt{x(2-3x)})}{2(1-6x)^2\,r^2} \tag{6.15}$$

and there is always a singularity at $r = 0$ that we classify as decompactification or conical singularity depending if the range of φ goes to ∞ or 0 (respectively) in the limit $r \to 0$.

In the table below we present the possible ranges for x, p, Λ, the sign of M and the singularity classification. The $\hat{\epsilon}$ refers to the relative sign between the gauge sector and the gravitational sector.

$\hat{\epsilon}$	x	p	Λ	M	singularity
$+1(ghosts)$	$]0,1/2[$	$]0,1/2[$	$]0,m^2/2[$	<0	*decomp.*
$-1(standard)$	$]0,1/6[$	$]-\infty,0[$	$]0,m^2/6[$	<0	*decomp.*
	$]1/6,1/2[$	$]1,+\infty[$	$]m^2/6,m^2/2[$	>0	*conical*

And to finalise we list the curvature sign and the existence or not of an horizon at

$r = 0$

$\hat{\epsilon}$	x	R	M	horizon
$+1(ghosts)$	$]0, 1/2[$	< 0	< 0	no
$-1(standard)$	$]0, 1/6[$	< 0	< 0	no
	$]1/6, 9/39[$	< 0	> 0	yes
	$]9/39, 9/26[$	> 0	> 0	yes
	$]9/26, 1/2[$	> 0	> 0	no

So we conclude that there is an horizon at $r = 0$ only for standard fields and the x range

$$x \in \left]\frac{1}{6}, \frac{9}{26}\right[$$

such that we obtain a dressed singularity. We note that the mass of the solution are positive in this range. All remaining cases hold a naked singularity.

6.2. Summary of Trivial Solutions

We will summarize only the results for null scalar field ($\phi = 0$), i.e. solutions without the scalar field at all. This case have been addressed in [2] without cosmological constant, we think is worthwhile to review these results with a non-null cosmological constant.

So, for $\phi = 0$ we found a electric extended configuration without singularities. The results are presented in terms of $K = \sqrt{m^2/2 - \Lambda}$ and the charge Q_e of the configuration. The metric and gauge field solutions for such configuration are

$$\begin{aligned} ds^2 &= \left(-\cosh^2(K\,r) + C_h^2(C_A\,\sinh(K\,r) + \theta)^2\right)\,dt^2 + dr^2 + C_h^2\,d\varphi^2 \\ &\quad +2C_h^2\,(C_A\,\sinh(K\,r) + \theta)\;dt\,d\varphi \\ A_0 &= \frac{C_E}{K}\sinh(K\,r) \end{aligned}$$

with

$$\begin{aligned}
K &= \sqrt{\frac{m^2}{2} - \Lambda} \\
C_h &= \frac{2Q_e}{\sqrt{\left|\frac{m^2}{2} + \Lambda\right|}} \\
C_A &= \frac{m}{2Q_e\,K}\sqrt{\left|\frac{m^2}{2} + \Lambda\right|} \\
C_E &= \frac{\mathrm{sign}\,(Q_e)}{2}\sqrt{\left|\frac{m^2}{2} + \Lambda\right|}
\end{aligned}$$

K can be both real and imaginary. In the case of imaginary K we obtain periodic solutions with period $2\pi/|K|$. We note that C_A is multiplying by $\sinh(K\,r)$ and correctly is also a pure imaginary such that $g_{t\varphi}$ is real.

The mass of these configurations is null and the angular momentum is

$$J = -\frac{4m\,Q_e^2}{\left|\frac{m^2}{2} + \Lambda\right|} \ .$$

There are no singularities and the curvature is constant

$$R = -K^2 + \Lambda = -\frac{m^2}{2} + 2\Lambda \ .$$

In the next table we list the ranges for K, Λ and the sign of the curvature

$\hat{\epsilon}$	Λ	K	R
$+1(ghosts)$	$]m^2, +\infty[$	$]-\infty, -m^2/2[$	< 0
$-1(standard)$	$]0, m^2[$	$]-m^2/2, 0[$	< 0
	$]0, +\infty[i$	$]0, m^2/4]$	≤ 0
	$]9/26, 1/2[$	$]m^2/4, +\infty[$	> 0

6.3. Discussion of Results

Given the Einstein Maxwell Chern-Simons theory coupled to a massless gravitational scalar field with action (2.1) discussed in section 2.1 we obtained the above classical solutions with electric charge only. We study both non-trivial and trivial solutions for the scalar field. For non-trivial solutions of the scalar field we obtain a rotating electric point particle that for the opposite sign between the gravitational and gauge sector and a certain range of the ratio Λ/m^2 is dressed, while for trivial solutions of the scalar field we find an extend charge configuration that cannot be interpreted as a particle.

For non-trivial solutions it turns out that the solutions are highly constraint depending on the cosmological constant to Chern-Simons coefficient squared $x = \Lambda/m^2$ which is constraint to the range $x \in]0, 1/2[$. Further requiring that the background obtained (in the limit $x \to 0$) to have no angular deficit we obtain only two free parameters, θ that accounts for the globally rotation of space and the ϕ exponential coefficient b. Both of them are not relevant for any physical observables. We study both non-trivial and trivial solutions for the scalar field. Also we consider both the cases for the relative sign between the gauge sector and the gravitational sector $\hat{\epsilon} = \pm 1$. When they have the same sign ($\hat{\epsilon} = +1$) we have that the gauge fields are ghosts in the sense that contribute a negative amount of energy to the Hamiltonian, while in the case that they have opposite sign ($\hat{\epsilon} = -1$) we have the standard case. Although the expressions for the solutions are expressed in the same way, the constants and consequently the physics change significantly. In particular the space-time curvature as well as the existence or non-existence of horizons will be sensitive to it.

For trivial solutions, the solutions are given in terms of $K = \sqrt{m^2/2 - \Lambda}$ and the charge Q_e and θ are free parameters. Although the cosmological constant is still bounded by the topological mass these bounds are not so restrictive. Again the relative sign $\hat{\epsilon} = \pm 1$ between the gravitational and gauge sector is relevant. In the limit $m \to 0$ we obtain for $\hat{\epsilon} = +1$ that $\Lambda < 0$ while for $\hat{\epsilon} = -1$ that $\Lambda > 0$. Our background is flat but with an angular deficit. In a similar way the curvature is sensitive to the relative sign $\hat{\epsilon}$.

The inclusion of the Chern-Simons topological term introduces very interesting features. Besides imposing the space to be rotating as explained in section 2.2 it imposes bounds on the cosmological constant trough the topological mass m. For the non-trivial solutions it constraints the allowed value for the cosmological constant to the interval $\Lambda \in]0, m^2/2[$ such that the limit $m \to 0$ corresponds also to $\Lambda \to 0$ (equivalent to $x \to 0$ and $p \to 0$) from the constraint $0 < \Lambda < m^2/2$ and we obtain in this limit a flat stationary background space-time. Then the cosmological constant is turn on and off by the Chern-Simons coefficient. It is very interesting that these facts emerges only as a consequence of the Chern-Simons term with out any ha-doc assumption. In this framework the existence of the cosmological constant can be interpreted as being due to the existence of the scalar field and the topological massive matter that constitute the electric point-particle. Therefore we can interpret that the charged matter deforms space-time such that the deformation is parameterized by the charge Q_e and Brans-Dicke coefficient λ and that the parameter x is given as a function of Q_e and λ. As expected this matter affects the curvature, either positively or negatively, depending on the sign of the gauge sector. For the trivial solution the cosmological constant bounds are not so restrictive but still exists a relation between topological mass and cosmological constant bounds, for $\hat{\epsilon} = +1$ we have $\Lambda < -m^2/2$ and for $\hat{\epsilon} = -1$ we have $\Lambda > -m^2/2$.

As already mentioned, for non-trivial solutions, we have that the cosmological constant is always positive. However concerning the curvature we have different behaviours depending on the relative sign between the gauge and gravitational sector.

For $\hat{\epsilon} = +1$ the curvature is always positive while for $\hat{\epsilon} = -1$ the curvature is positive only for high values of $x = \Lambda/m^2 \in]9/38, 1/2[$. To understand why let us contract the Einstein equations with the metric such that we obtain the relation

$$R = 3e^{b\phi}\Lambda - \lambda(\partial\phi^2) + \hat{\epsilon}e^{c\phi}E^2 \ .$$

For solutions with $\hat{\epsilon} = +1$ the Brans-Dicke coefficient is always negative, hence all terms contribute positively to the curvature. For solutions with $\hat{\epsilon} = +1$ we have that the electric field contribution is always negative and that the Brans-Dicke coefficient is positive when $x \in]0, 1/6[$ and negative when $x \in]1/6, 1/2[$. Therefore we have the following cases, for $x \in]0, 1/6[$ both the scalar field and electric field contribute negatively to the curvature while for $x \in]1/6, 1/2[$ the scalar field contributes positively and the electric field contributes negatively to the curvature. We further note that from the expressions for the several constants (3.25) for the allowed solutions (3.24), we have that near $x = 1/6$ the electric field contribution is predominant when compared with the scalar field contribution (that is negletable, $C_\phi \to 0$). So only away from $x = 1/6$ the scalar matter will become dominant over the charged matter and we have a positive curvature for $x \in]9/38, 1/2[$. In this way we conclude that the scalar field is determinant in imposing the bounds on the cosmological constant (on the non-trivial solutions). Also it is the scalar field that allows for the existence of horizons. We concluded that there are horizons only for $\hat{\epsilon} = -1$ in the range $1/6 < x < 9/26$ which corresponds to the greater positive values of p $(> 3/2)$, remembering that the Brans-Dicke coefficient is proportional to $\lambda \sim p$ this means that these values correspond to a region in which the scalar matter contributes positively to the curvature.

For the trivial solutions although the bound on the cosmological constant is not so restrictive the same behaviour concerning the curvature applies as can be seen directly in the expression for the curvature (4.14) that depends both in the cosmological constant and topological mass. For the trivial solutions we will have positive curvature only for $\hat{\epsilon} = -1$ and $\Lambda > m^2/2$.

The charges and angular momenta of the configurations are finite. The solutions are, for both non-trivial and trivial solutions of the scalar field, rotating spaces with angular momenta $J \sim m$ (or $J \sim \ \mathrm{sign}\,(m)$), this accounts explicitly for the known parity P and time-inversion T violation due to the Chern-Simons term [23]. Is explicit in the sense that the sign of the constant C_A and of the angular momentum only depends on the relative sign between the Chern-Simons coefficient and the gravitational curvature term.

Concerning the mass of our configurations we concluded that its positiveness (or negativeness) is sensitive to the relative sign between the gravitational and gauge sector. However these results are not conclusive, although the charge and angular momentum are finite, the mass is infrared divergent, this is the main drawback of our solutions. The background is flat and therefore the reference mass (of the background) is null. Here we consider a cut-off of the order of the Planck Length. We believe that something is still missing in our theory, as already explained previously we are not considering a

gravitational Chern-Simons. This correction to the Einstein action induces a correction to the configuration mass and would regularize it [41, 42, 43, 44, 45, 46]. For the extended trivial solutions the mass is null.

As a final remark we note that our solutions hold that $a = 0$. Therefore the gravitational sector resembles an action with a dilatonic potential given by our cosmological constant term in (2.1), see for instance [14, 15, 50]. We notice that by setting $a = 0$ the field ϕ is only minimally coupled to the $2+1$ metric and all the fields are expressed in terms of the scalar field (see the derivation of the solutions in section 3), therefore we would expect to obtain similar results by including more generic dilatonic potentials. An important point to stress here is that although our action is similar to the action of the work of Chan and Mann [14, 15] (CM) with an extra Chern-Simons term, it is not possible to obtain the solutions of those works in the limit $m \to 0$. The main reason is that the although there the CM action is generic the authors only consider solutions for the particular case in which the scalar field can be interpreted as a dilaton. This means that the constants in our action (2.1) would be $a = 0$, $c = -b = 4$ and $\lambda = 8$ which is not the case since our constants are related as $c = -b/2$ and λ is dependent on several parameters. Therefore our massless scalar field cannot be interpreted as a dilaton. Secondly in our case we have no horizons away from $r = 0$ and both our cosmological constant Λ and charge Q_e vanishes in the limit $m \to 0$ as already explained in detail, therefore we cannot possible obtain the solutions of Chan and Mann since their horizons are set uniquely by Λ and Q_e.

Interesting enough our gravitational field ϕ can be related to the works of polarized cylindrical gravitational waves in $3+1$ gravity [27, 28, 30, 31]. For an explicit form of the effective $2+1$ dimensional action see for instance equation (1) of [31] (see also [30]). In our case we further have a full gauge sector such that our classical solutions could constitute a possible electric charged background with cilindrical symmetry in $3+1$ dimensions (our solutions would correspond then to a electric charged string). Also similar actions have been considered in cosmological scenarios [52] and in brane worlds [53].

After finishing this work the author realized that after we get our solutions redefining the radial coordinate accounts for changing the dilaton coupling (for $a \neq 0$) with the curvature R and the Brans-Dicke parameter, however they will have generally different exponential factors, this does not invalidate the work presented here, simply we could yet consider a more generic action.

As an extension to this work the author intends to compute a pure magnetic solution [51] using a similar action and procedure to this article. In order such configuration to exist it is necessary to consider an external electric charge distribution because as can be seen explicitly from Maxwell equations (2.2) or (A.25) for $E = E_* = 0$ we have that $B \sim j^0$. If we set $j^0 = 0$ the equations of motion hold that the magnetic field is null. This discussion is already put forward by Kogan [2] (see conclusions of this reference). Another possible way out is to consider $Ef = hAB$ (such that $E_* = 0$, see (A.18) in appendix). In these cases the rotation will induce a electric field (see

discussion in section 2.3). Also as other possible direction of research it would be interesting to consider extensions of this work that include gravitational Chern-Simons (as already explained we would expect to obtain finite mass) and dilatonic potentials.

Acknowledgments

This work was supported by Grant SFRH/BPD/5638/2001, SFRH/BPD/17683/2004 and POCTI/P-FIS-57547/2004. The author thanks both Ian Kogan for pointing out [2] and Bayram Tekin for suggesting to further investigate gravitational solutions of Chern-Simons theories. The author thanks José Sande Lemos and Óscar Dias for invaluable discussions, also for explaining in detail part of their works [17, 18, 19, 20] as well recommending literature on the subject. The author thanks Nuno Reis and Carlos Herdeiro for several discussions and suggestions. The author thanks Paulo Vargas Moniz and Dmitri Gal'tsov for important remarks and suggestions.

Appendix A. Cartan Formalism

In this appendix we study the equations of motion in the Cartan Frame.

The Lagrangean 3-form corresponding to the action (2.1) is rewritten as

$$\begin{aligned} \mathcal{L} = \ & -\Big\{ e^{a\phi}\left[R*1+2\lambda\, d\phi\wedge *d\phi\right] - e^{b\phi}\Lambda * 1 \\ & +\hat{\epsilon} e^{c\phi}\left[F\wedge *F + *J\wedge A\right] + \hat{\epsilon}\tfrac{m}{2}\, A\wedge F\Big\} \end{aligned} \tag{A.1}$$

with R the metric curvature and $F = dA$ and where we define the Hodge dual as usual

$$(*X)^{i_1\ldots i_q} = (-1)^D \frac{\sqrt{-g}}{p!}\epsilon^{i_1\ldots i_q\, j_1\ldots j_p} X_{j_1\ldots j_p} \tag{A.2}$$

Introducing a triad $\{e^0, e^1, e^2\}$ such that

$$e^i = e^i{}_\alpha dx^\alpha \qquad g_{\alpha\beta} = \eta_{ij} e^i{}_\alpha e^j{}_\beta \tag{A.3}$$

where the Greek indices refer to the coordinates $(x^0 = t, x^1 = r, x^2 = \varphi)$ and the roman ones to the Cartan frame triad (meaning the flat space indices).

Varying the Lagrangean with respect to the Gauge field A, the coframe field e^i and the dilaton ϕ we obtain the equations of motion in the Cartan frame

$$\begin{aligned} d(*F\, e^{c\phi}) - *J &= -\frac{m}{2}\, F \\ \left[e^{a\phi}\left(G_{ij}+\Phi_{ij}\right) - e^{b\phi}\eta_{ij}\Lambda * e_i - 2e^{c\phi}T_{ij}\right] * e^j &= 0 \\ e^{a\phi}\left[(4a^2-\lambda)d*d\phi + a(4a^2-2\lambda)d\phi\wedge *d\phi\right] & \\ -(b-3a)e^{b\phi}\Lambda * 1 &= \hat{\epsilon} 2(a+c)e^{c\phi} F\wedge *F \end{aligned} \tag{A.4}$$

respectively the Maxwell, Einstein and scalar field equations. We will specify the Einstein tensor G_{ij}, the Energy-Momentum tensor F_{ij} and the scalar field tensor Φ_{ij} for each metric parameterization used

$$\begin{aligned} G_{ij} &= R_{ij} - \tfrac{1}{2}\eta_{ij} R \\ T_{ij} &= \hat{\epsilon}\left(F_{ik}F_j{}^k - \frac{1}{4}\eta_{ij}F_{kl}F^{kl}\right) \\ \Phi_{ij} &= -a\nabla_i\partial_j\phi + a\,\eta_{ij}\nabla^2\phi + (\lambda - a^2)\partial_i\phi\partial_j\phi - \left(\frac{\lambda}{2} - a^2\right)\eta_{ij}\partial_k\phi\partial^k\phi \end{aligned} \tag{A.5}$$

To proceed further one has to introduce a spin connection $\omega^{ij}{}_\alpha$ and define the corresponding connection 1-form

$$\omega^i{}_j = \omega^i{}_{j\alpha} dx^\alpha = \omega^i{}_{jk} e^k \tag{A.6}$$

Using the antisymmetric property (from definition)

$$\omega^{ij} = -\omega^{ji} \tag{A.7}$$

and the Cartan Structure equation

$$de^i + \omega^i{}_j \wedge e^j = 0 \tag{A.8}$$

is enough to determine all the connection coefficients ${\omega^i}_{jk}$. In this work we are considering only radial symmetric configurations and metric parameterization such that $e^1 = dr$ (note that a redefinition of r introduces a non trivial metric component g_{11}) and e^0 and e^2 depend only on dt and $d\varphi$ (means that the metric has nonnull components $g_{\alpha\alpha}, g_{02}$). In these particular cases we get the non vanishing connection coefficients

$$\begin{gathered} {\omega^0}_{12} = {\omega^1}_{02} = {\omega^1}_{20} = -{\omega^2}_{10} \qquad {\omega^0}_{21} = {\omega^2}_{01} \\ {\omega^0}_{10} = {\omega^1}_{00} \qquad {\omega^0}_{20} = {\omega^2}_{00} \\ {\omega^0}_{22} = {\omega^2}_{02} \qquad {\omega^1}_{22} = -{\omega^2}_{12} \end{gathered} \tag{A.9}$$

plus the two equations

$$\begin{gathered} de^0 + {\omega^0}_1 \wedge e^1 + {\omega^0}_2 \wedge e^2 = 0 \\ de^2 + {\omega^2}_0 \wedge e^0 + {\omega^2}_1 \wedge e^1 = 0 \end{gathered} \tag{A.10}$$

Also note that in this case the only Electric field component is $E = F_{01}$ ($F_{02} = 0$ from Maxwell equations) and all the derivatives are with respect to r only. Then it is now possible to define T_{ij} and Φ_{ij} for our parameterization:

$$\begin{aligned} 2T_{00} &= \hat{\epsilon}\,(B^2 + E^2) \\ 2T_{11} &= \hat{\epsilon}\,(B^2 - E^2) \\ 2T_{22} &= \hat{\epsilon}\,(B^2 + E^2) \\ 2T_{02} &= -2\hat{\epsilon} BE \end{aligned} \tag{A.11}$$

the square of the Maxwell tensor is

$$F^2 = 2\hat{\epsilon}(B^2 - E^2) \tag{A.12}$$

and

$$\begin{aligned} \Phi_{00} &= -a\phi'' + (\lambda/2 - a^2)(\phi')^2 \\ \Phi_{11} &= \lambda/2(\phi')^2 \\ \Phi_{22} &= a\phi'' - (\lambda/2 - a^2)(\phi')^2 \end{aligned} \tag{A.13}$$

Note that the original electric field $E_{*\,\alpha} = F_{t\alpha}$, magnetic field $B_* = F_{r\varphi}$ and external current $*J$ are related to the Cartan frame ones $E_i = F_{0i}$, $B = F_{12}$ and $*j$ either by using the triad e^i_α or by the definition of the 2-forms $F = F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta = F_{ij}\, e^i \wedge e^j$ and $*J = \sqrt{-g}\,\epsilon_{\mu\nu\rho}\, J^\mu\, dx^\nu \wedge dx^\rho = \epsilon_{ijk}\, j^i\, e^j \wedge e^k$.

We use the metric parameterization such that the line element is given by

$$ds^2 = -f^2 dt^2 + dr^2 + h^2(d\varphi + Adt)^2 \tag{A.14}$$

such that the usual components read

$$\begin{aligned} g_{00} &= -f^2 + h^2 A^2 \\ g_{11} &= 1 \\ g_{22} &= h^2 \\ g_{02} &= h^2 A \end{aligned} \tag{A.15}$$

The Cartan triad is then given by

$$\begin{array}{lllllll} \mathrm{e}^{\,0} = d\theta^0 & = & f dt & \qquad e^0{}_0 = f & e^0{}_1 = 0 & e^0{}_2 = 0 \\ \mathrm{e}^{\,1} = d\theta^1 & = & dr & \qquad e^1{}_0 = 0 & e^1{}_1 = 1 & e^1{}_2 = 0 \\ \mathrm{e}^{\,2} = d\theta^2 & = & h(d\varphi + A dt) & \qquad e^2{}_0 = hA & e^2{}_1 = 0 & e^2{}_2 = h \end{array} \tag{A.16}$$

such that the line element is now

$$ds^2 = e^i e_i = \eta_{ij} d\theta^i d\theta^j = -(d\theta^0)^2 + (d\theta^1)^2 + (d\theta^2)^2 \tag{A.17}$$

with Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1)$.

The original Electric E_* and Magnetic fields B_* are given by

$$\begin{array}{lll} E_* & = & E\,f - B\,h\,A \\ B_* & = & B\,h \end{array} \tag{A.18}$$

where E and B are the Cartan frame Electric and Magnetic fields.

The Cartan external currents j^i are given by

$$\begin{array}{lllll} j^0 & = & f\,J^t & = & \dfrac{e^{-c\phi}}{h}\mathcal{J}^t \\ j^2 & = & h\left(J^\varphi - A\,J^t\right) & = & \dfrac{e^{-c\phi}}{f}\left(\mathcal{J}^\varphi - A\,\mathcal{J}^t\right) \end{array} \tag{A.19}$$

where J^μ are the original external currents. For radial currents one has simply $j^1 = J^1 = e^{-c\phi}\mathcal{J}^r/hf$. We note that in terms of the physical $\mathcal{J}^\mu$ (measured by an external observer) we have $J^\mu = e^{c\phi}\mathcal{J}^\mu/hf$ (see eq (2.20)).

From the form differentials

$$\begin{array}{lll} de^0 & = & -\alpha e^0 \wedge e^1 \\ de^2 & = & \beta e^1 \wedge e^2 - \gamma e^0 \wedge e^1 \end{array} \tag{A.20}$$

we conclude that, except for the external currents, the Equations of motion, connections, curvature and so on depend only on the combinations

$$\alpha = \frac{f'}{f} \qquad \beta = \frac{h'}{h} \qquad \gamma = \frac{h\,A'}{f} \tag{A.21}$$

We list the non null connections in the Cartan frame

$$\begin{array}{l} \omega^0{}_{10} = \omega^1{}_{00} = \alpha \\ \omega^0{}_{12} = \omega^0{}_{21} = \omega^1{}_{02} = \omega^1{}_{20} = \omega^2{}_{01} = -\omega^2{}_{10} = -\gamma/2 \\ \omega^1{}_{22} = -\omega^2{}_{12} = -\beta \end{array} \tag{A.22}$$

and the Einstein tensor components

$$\begin{aligned} G_{00} &= -\beta^2 - \gamma^2/4 - \beta' \\ G_{11} &= \alpha\beta + \gamma^2/4 \\ G_{22} &= \alpha^2 - 3\gamma^2/4 + \alpha' \\ G_{02} &= -\beta\gamma - \gamma'/2 \end{aligned} \tag{A.23}$$

Then the Maxwell Equations are

$$B' + \alpha B + c\,B\,\phi' - \gamma\,E - j^2 = -\,m\,E\,e^{-c\phi} \tag{A.24}$$

$$E' + \beta E + c\,E\,\phi' + j^0 = -\,m\,B\,e^{-c\phi} \tag{A.25}$$

The Einstein Equations are

$$e^{a\phi}\left(\beta\gamma + \frac{\gamma'}{2}\right) = 2\hat{\epsilon}e^{c\phi}E\,B \tag{A.26}$$

$$e^{a\phi}\left[\beta^2 + \frac{\gamma^2}{4} + \beta' + a\phi'' + \left(a^2 - \frac{\lambda}{2}\right)(\phi')^2\right] + \frac{1}{2}e^{b\phi}\Lambda = -\,\hat{\epsilon}(B^2+E^2)e^{c\phi} \tag{A.27}$$

$$e^{a\phi}\left[\alpha^2 - \frac{3\gamma^2}{4} + \alpha' + a\phi'' + \left(a^2 - \frac{\lambda}{2}\right)(\phi')^2\right] + \frac{1}{2}e^{b\phi}\Lambda = \hat{\epsilon}(B^2+E^2)e^{c\phi} \tag{A.28}$$

$$e^{a\phi}\left[\alpha\beta + \frac{\gamma^2}{4} + \frac{\lambda}{2}(\phi')^2\right] + \frac{1}{2}e^{b\phi}\Lambda = \hat{\epsilon}(B^2-E^2)e^{c\phi} \tag{A.29}$$

and the dilaton equation is

$$e^{a\phi}\left[(4a^2-\lambda)\phi'' + a\left(4a^2-2\lambda\right)(\phi')^2\right] + (3a-b)e^{b\phi}\Lambda = \hat{\epsilon}(a+c)(B^2-E^2)e^{c\phi} \tag{A.30}$$

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